A numerical method for time-dependent anisotropic eikonal equations

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In this note we consider the first order scalar PDE

$$H(x, u, \nabla u) = \|A(x, u)\nabla u\| - 1 = 0, \quad x \in \Omega \subset \mathbb{R}^n, \tag{1}$$

with the initial condition

$$u(x_0) = 0, \tag{2}$$

where $(n \times n)$ matrix $A(\cdot, \cdot)$ satisfies the inequality

$$\det A(x, u) > \eta > 0 \quad \forall x \in \Omega, \quad u \geq 0.$$

Though the above equation contains no time variable, it can be viewed as an anisotropic "time-dependent" eikonal equation. Indeed, one may show that any map $u = u(x)$, whose sublevel sets $\{x \mid u(x) \leq t\}$ coincide with the reachable sets $\mathcal{R}(t)$ of the differential inclusion

$$\begin{cases} \dot{x} \in [A(x, t)]^{-1} B, \\ x(0) = x_0, \end{cases} \qquad B = \{v \mid \|v\| \leq 1\},$$

must be a viscosity solution of the problem (1), (2).

The proposed numerical scheme is based on the Fast Marching Method (FMM) [1,2], which proved to be very effective for solving classical eikonal equations. Note that the convergence of the finite difference scheme for the problem (1), (2) requires a monotonicity of the hamiltonian [3]

$$\frac{\partial H}{\partial u} > 0, \quad u > 0.$$

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