Cutting plane method using penalty functions

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We propose a method of solving a convex programming problem. It is based on the ideas of cutting methods (e.g., [1]) and penalty methods (e.g., [2]). The method uses the polyhedral approximation of the feasible set and the auxiliary functions epigraphs that are constructed on the basis of external penalties.

We solve the problem $\min\{f(x) : x \in D\}$, where $f(x)$ — a convex function in R_n and $D \subset R_n$ — convex bounded closed set.

Let $f^* = \min\{f(x) : x \in D\}$, $X^* = \{x \in D : f(x) = f^*\}$, $\mathrm{epi}\,(g, G) = \{(x, \gamma) \in R_{n+1} : x \in G, \gamma \geqslant g(x)\}$, where $G \subset R_n$, $g(x)$ — the function defined in R_n, $W(z, Q)$ — the bunch of normalized generally support vectors for the set Q at the point z, $\mathrm{int}\,Q$ — interior of the set Q, $K = \{0, 1, \ldots\}$.

We set $F_i(x) = f(x) + P_i(x)$, $i \in K$, where $P_i(x)$ — a penalty function that satisfies the conditions:

$$P_i(x) = 0\,\forall\, x \in D,\ P_{i+1}(x) \geqslant P_i(x),\ \lim_{i \in K} P_i(x) = +\infty \text{ for all } x \notin D. \tag{1}$$

The proposed method generates a sequence of approximations $\{x_k\}$ as follows. Fix a number $\varDelta_0 > 0$, a point $v = (v', \gamma')$, where $v' \in \mathrm{int}\,D$, $\gamma' > f(v')$ and define a convex penalty function $P_0(x)$ with the condition (1). Construct convex closed sets $M_0 \subset R_{n+1}$ and $D_0 \subset R_n$ such that $\mathrm{epi}\,(F_0, R_n) \subset M_0$, $D \subset D_0$. Set $\bar{\gamma} \leqslant f_0^*$, where $f_0^* = \min\{f(x) : x \in D_0\}$. Fix $i = 0$, $k = 0$.

1. Find $u_i = (y_i, \gamma_i)$, where $y_i \in R_n$, $\gamma_i \in R_l$, as a solution of the problem
   $$\min\{\gamma : x \in D_i,\ (x, \gamma) \in M_i,\ \gamma \geqslant \bar{\gamma}\}.$$
   If $u_i \in \mathrm{epi}\,(f, D)$, then $y_i \in X^*$.
2. In some way choose a point $v_i \notin \mathrm{int}\,\mathrm{epi}\,(F_i, R_n)$ in the interval (v, u_i). Let $M_{i+1} = M_i \cap \{u \in R_{n+1} : \langle a_i, u - v_i\rangle \leqslant 0\}$, where $a_i \in W(v_i,\ \mathrm{epi}\,(F_i, R_n))$.
3. Let $D_{i+1} = D_i \cap \{x \in R_n : \langle b_i,\ x - v_i'\rangle \leqslant 0\}$, where $b_i \in W(v_i', D)$, $v_i' = (v', y_i) \setminus \mathrm{int} D$, Or $D_{i+1} = D_i$.
4. If $F_i(y_i) - \gamma_i > \varDelta_k$, than set $P_{i+1}(x) = P_i(x)$ and go step 5. Otherwise choose convex penalty function $P_{i+1}(x)$ satisfying the given conditions (1) and set $x_k = y_i$, $\sigma_k = \gamma_i$. Choose $\varDelta_{k+1} > 0$ and go to step 5 with the value of k increased by one.
5. Increase the value of i by one and go to step 1.

Lets note that it is possible to set $\varDelta_k = \varDelta > 0$, for all $k \in K$, and, in particular, suppose that $\varDelta$ is arbitrarily large or set $\varDelta_k \to 0$, $k \to \infty$, $k \in K$.

It is proved that for every limit point $(\bar{x}, \bar{\sigma})$ of the sequence $\{(x_k, \sigma_k)\}$ the following equalities hold:

$$\bar{x} \in X^*, \quad \bar{\sigma} \in f^*.$$

References

1. Bulatov, V. P.: Embedding methods in optimization problems (in Russian), p. 161. Nauka, Novosibirsk (1977)
2. Vasilev, F. P.: Optimization methods (in Russian), p. 620. MCCME, Moscow (2011)