The Longest Vector Sum Problem: Complexity and Algorithms

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In the *longest vector sum problem* (LVS), we are given a set X of n vectors in a normed space $(\mathbb{R}^d, \|.\|)$, the goal is to find a subset $S \subseteq X$ maximizing the value of $\big\| \sum_{x \in S} x \big\|$. The variation where the subset S is required to have a given cardinality $k \in [1, n]$ will be called the *longest k-vector sum problem* (Lk-VS).

Both of these problems are strongly NP-hard if the Euclidean norm is used [4, 1] and polynomially solvable when d is fixed. The best known algorithm for the general LVS runs in time $O(n^d)$ [3], the Lk-VS can be solved in time $O(n^{4d})$ [2]. In the Euclidean case, both problems admit an FPTAS for any fixed d [1].

Our Contributions. First, we prove that, for any ℓ_p norm, $p \in [1, \infty)$, the LVS and Lk-VS problems are hard to approximate within a factor better than $\min\{\alpha^{1/p}, \sqrt{\alpha}\}$, where α is the inapproximability bound for Max-Cut, $\alpha = 16/17$ (or $\alpha \approx 0.879$ if the Unique Games Conjecture holds).

On the other hand, for an arbitrary norm, we reduce the computational time for both problems. We propose an $O(dn^{d-1} \log n)$-time algorithm for the LVS and an $O(dn^{d+1})$-time algorithm for the Lk-VS. In particular, the two-dimensional LVS problem can be solved in a nearly linear time.

Finally, for any norm, we propose a randomized algorithm which finds $(1-\varepsilon)$-approximate solutions of both problems in time $O(d\,(1+\frac{2}{\varepsilon})^d\, n)$. In particular, we have a linear-time algorithm for any fixed d and ε. The algorithm is polynomial for instances of dimension $O(\log n)$.

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