DUAL ALGORITHMS for LINEAR SEMIDEFINITE OPTIMIZATION

Vitaly Zhadan

Dorodnicyn Computing Centre, FRC CSC, Vaviliva st. 40, Moscow, 119333, Russia

In this paper we consider dual affine-scaling and simplex-like algorithms for linear semidefinite optimization. The main attention will be given to the combined algorithms with both properties of affine-scaling and simplex-like methods.

Let $\mathbb{S}^n$ denote the space of real symmetric matrices of order n, and let $\mathbb{S}^n_+$ denote the cone of positively semidefinite matrices from $\mathbb{S}^n$. The scalar (inner) product $M \bullet M_2$ of two matrices M_1 and M_2 from $\mathbb{S}^n$ is defined as the trace of the matrix $M_1 M_2$.

The linear semi-definite programming problem is to find

$$\min_{X \in \mathcal{F}_P} C \bullet X, \qquad \mathcal{F}_P = \left\{ X \in \mathbb{S}^n_+ : \ A_i \bullet X = b^i, \quad i = 1, \ldots, m \right\}, \tag{1}$$

where $C \in \mathbb{S}^n$ and $A_i \in \mathbb{S}^n$, $1 \leq i \leq m$, are given. The dual problem to (1) has the form

$$\max_{u \in \mathcal{F}_D} b^T u, \qquad \mathcal{F}_D = \left\{ u \in \mathbb{R}^m : \ V(u) = C - \sum_{i=1}^{m} u^i A_i \in \mathbb{S}^n_+ \right\}, \tag{2}$$

where $b = [b^1, \ldots, b^m]$. We assume that the matrices A_i, $1 \leq i \leq m$, are linear independent. We assume also that problems (2) is nondegenerate, i.e. all feasible points from $\mathcal{F}_D$ are nondegenerate.

Let X_* and $[u_*, V_*]$ with $V_* = V(u_*)$ be the solutions of problems (1) and (2) respectively. In what follows we assume that the strict complementary condition holds in $[X_*, V_*]$.

At first consider the simplex-like algorithm. Denote by $\mathcal{E}(\mathcal{F}_D)$ the subset of extreme points of the feasible set $\mathcal{F}_D$. Starting from the extreme point u_0 the algorithm generates the sequence of extreme points $\{u_k\} \subset \mathcal{E}(\mathcal{F}_D)$ which converges to u_*. The optimality conditions for both problems (1) and (2) are used essentially to make pivoting at each step of the algorithm.

The second algorithm is the combination of the affine-scaling and simplex-like algorithms. At each iteration the search direction Δu in this algorithm is the linear combination of two directions, namely, $\Delta u = \Delta u^1 + \tau \Delta u^2$, where τ is a positive coefficient. The direction Δu^1 belongs to the minimal face of the set $\mathcal{F}_D$, containing the current point u. This direction is defined in the usual way by means of approach using in dual affine-scaling algorithms. The second direction Δu^2 gives us possibility to jump from one boundary face to another one. In the case where the minimal face is an extreme point of $\mathcal{F}_D$, the method can behave as the simplex method. Both algorithms converge to the solution u_* of the dual problem (2). Simultaneously we obtain the solution X_* of the problem (1).